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Computation of Critical Exponents for Two-Dimensional Ising Model on a Cellular Automaton

B. Kutlu[1] **and N. Aktekin**[1]

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Static critical exponents for the two-dimensional Ising model are computed on a cellular automaton. The analysis of the data within the framework of the finite-size scaling theory reproduces their well-established values.

KEY WORDS: Ising model; cellular automata; critical exponents.

1. INTRODUCTION

Although the conventional Monte Carlo method is the most extensively used method for Ising model investigations, a faster method would allow the computations to be extended to larger lattices. The algorithm due to Creutz[(1)] for the simulation of Ising model on a cellular automaton does not require high-quality random numbers, and it is an order of magnitude faster than the conventional Monte Carlo method. Compared to the Q2R cellular automaton model,[(2)] this algorithm has the advantage of fluctuating internal energy which permits the computation of the specific heat in the same way as the susceptibility is computed from the fluctuations of the order parameter. The values of the static critical exponents for the two-dimensional Ising model are well established,[(3)] and in order for this method to be considered successful, it must reproduce these well-established values. For this purpose, the static critical exponents α, β, and γ, which are related to the specific heat, the order parameter, and the susceptibility, are computed by using the algorithm due to Creutz. The details of the model are given in Section 2, the data are analyzed and the results are discussed in Section 3, and a conclusion is given in Section 4.

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[1] Gazi Üniversitesi, Fen-Edebiyat Fakültesi, Fizik Bölümü, Teknikokullar, 06500 Ankara, Turkey.

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2. MODEL

Four binary bits are associated with each site of the lattice. The value of each site is determined from its value and those of its nearest neighbors at the previous time step. The updating rule, which defines a deterministic cellular automaton, is as follows. Of the four binary bits on each site, the first one is the Ising spin B_i. Its value may be 0 or 1. The Ising spin energy (internal energy) H_I is given by

$$H_I = -J \sum_{\langle ij \rangle} S_i S_j \tag{1}$$

where J is the nearest neighbor coupling constant, $S_i = 2B_i - 1$, and $\langle ij \rangle$ denotes the sum over all nearest neighbor pairs of sites. The second and the third bits are for the momentum variable conjugate to the spin (the demon). These two bits form an integer which can take the value 0, 1, 2, or 3. The kinetic energy associated with the demon H_K can take on four times these integer values. The total energy H,

$$H = H_I + H_K \tag{2}$$

is conserved. For a given total energy the system temperature is obtained from the average value of the kinetic energy. The fourth bit provides a checkerboard-style updating, and so it allows the simulation of the Ising model on a cellular automaton. The black sites of the checkerboard are updated and then their color is changed into white; white sites are changed into black without being updated.

The updating rules for the spin and the momentum variables are as follows: For a site to be updated its spin is flipped and the change in the Ising energy (internal energy) H_I is calculated. If this energy change is transferable to or from the momentum variable associated with this site such that the total energy H is conserved, then this change is done and the momentum is appropriately changed. Otherwise the spin and the momentum are not changed.

For the initial values all the spins are taken ordered (up or down) and the kinetic energy is given to the lattice via the second bits of momentum variables in the white sites randomly.

The quantities computed are averages over the lattice and the number of time steps during which the cellular automaton develops.

3. RESULTS AND DISCUSSION

The simulations are done on square lattices ($L \times L$) with periodic boundary conditions. For each lattice its critical temperature $T_c(L)$ is determined from the maxima of both the specific heat C and magnetic susceptibility χ. For each run 50,000 time steps are enough for taking measurements to be used for computing equilibrium quantities.[4] The data are analyzed within the framework of the finite-size scaling theory.[5]

The finite-size scaling relation for the size-dependent shifting of the specific heat maxima $T_c(L) - T_c(\infty) \propto aL^{-1/\nu}$, with $\nu = 1$, is verified by the computed data and the straight line which fits to the data gives, when extrapolated to $1/L \to 0$, $T_c^C(\infty) = 2.263$. The corresponding scaling relation for the susceptibility maxima is also verified by the data and gives the same value of $T_c^\chi(\infty) = 2.263$ within the error limits. These values are in agreement with the theoretical prediction of $T_c(\infty) = 2.269$.

The data obtained for the order parameter M are analyzed by making use of the finite-size scaling plot given in Fig. 1. The data lie on a single curve for temperatures both above and below $T_c(\infty) = 2.263$, and validate the finite-size scaling. For large L and so for large values $x = \varepsilon L^{1/\nu}$, the infinite lattice critical behavior must be asymptotically reproduced, that is,

$$X(x) \propto Bx^{\beta} \tag{3}$$

for very large x. The straight line passing through the data for $T < T_c(\infty)$ in Fig. 1 describes Eq. (3). The straight line passing through the data for

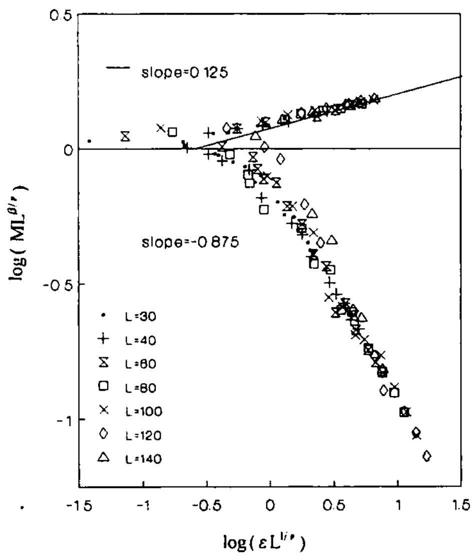

Fig. 1. Finite-size scaling plot for the order parameter M for $T < T_c(\infty)$ ($\beta = 0.125$) and for $T > T_c(\infty)$ ($\beta' = 0.875$); $\varepsilon = [T - T_c(\infty)]/T_c(\infty)$.

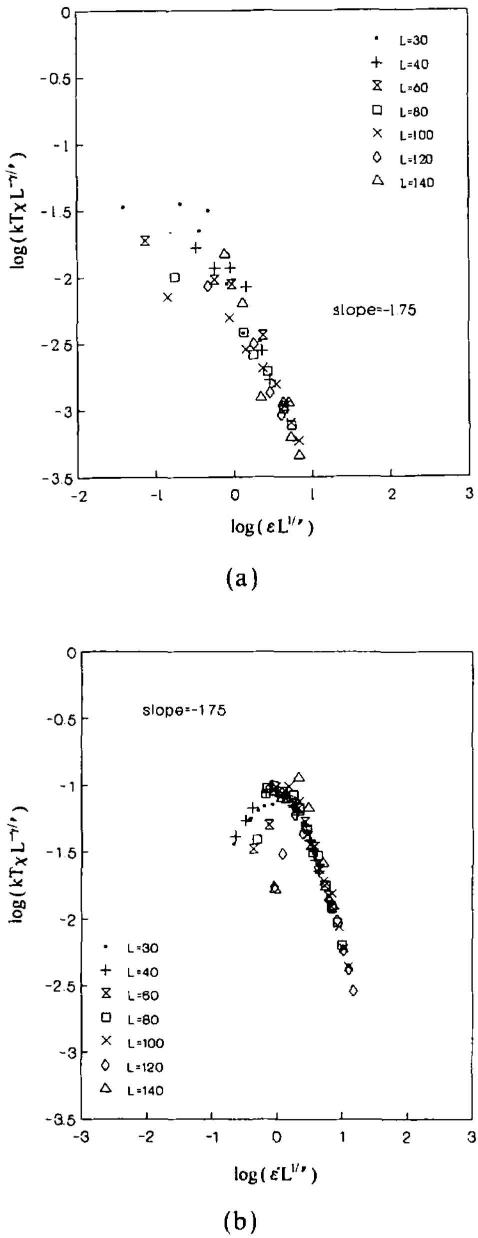

Fig. 2. Finite-size scaling plot for the susceptibility χ. (a) $T < T_c(\infty)$, $\varepsilon = [T - T_c(\infty)]/T_c(\infty)$; (b) $T > T_c(\infty)$, $\varepsilon' = [T - T_c(\infty)]/T$.

$T > T_c(\infty)$ behaves according to this equation with $\beta' = 1 - \beta$ replacing β and some other constant replacing B. Thus, the data for M are in agreement with the theoretical value $\beta = 0.125$ for $T < T_c(\infty)$ and $\beta' = 0.875$ for $T > T_c(\infty)$.

Finite-size scaling plots of the susceptibility are shown in Fig. 2 together with the straight lines describing the theoretically predicted behavior for large x,

$$Y(x) \propto Gx^{-\gamma} \tag{4}$$

The scaling of the susceptibility data agrees well with the asymptotic form, and so with the critical exponent $\gamma = 1.75$ for both $T > T_c(\infty)$ and $T < T_c(\infty)$.

To get another estimation for these critical exponents, the finite-size scaling relations at $T_c(\infty)$ are used. Since the temperature is a computed quantity in the present algorithm, getting values of the quantities at a given temperature is not straightforward. This difficulty is overcome in either of the following ways: (a) If the curve for the temperature dependence of the quantity is smooth enough, the data are interpolated between two values at temperatures above and below the given temperature, assuming that the temperature dependence within the small interval (~ 0.01) between these two values is linear. (b) If the curve for the quantity is not smooth enough, an average value is obtained by using the values of this quantity within an interval (~ 0.005) above this temperature; similarly, an average value below this temperature is found. The average of these two values is assigned to the quantity as its value at the given temperature. These two methods are applied for getting values for M at $T_c(\infty)$. The slope obtained from the log–log plot of the scaling relation corresponding to this quantity gives $\beta/\nu = 0.135$ for method (a), and method (b) results in $\beta/\nu = 0.123$. As is seen from these values, the result given by method (b) is in very good agreement with the theoretical value $\beta/\nu = 0.125$. These two values also indicate that the computed curve for M is not smooth enough to apply the interpolation. Since the curves for the temperature dependence of χ and C are not smooth enough, method (b) is adopted for getting their values at $T_c(\infty)$ in plotting the respective scaling relations corresponding to these quantities. From the slopes of the straight lines the following values are obtained: $\gamma/\nu = 1.89$ and $\alpha/\nu = 0.064$. Both of these values are higher than the theoretical values, $\gamma/\nu = 1.75$ and $\alpha/\nu = 0$ (log), respectively. There is an alternative expression for computing χ. This results from the fact that for $T \geqslant T_c(\infty)$, $\langle M \rangle$ vanishes, and thus χ involves only $\langle M^2 \rangle$. The log–log plot of these data against L gives $\gamma/\nu = 1.75$. This agrees very well with the theoretical prediction.

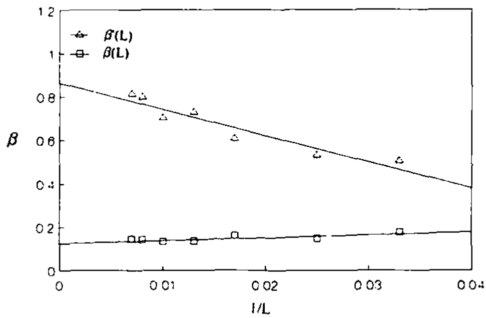

Fig. 3. Variation of the critical exponent $\beta(L)$ for the order parameter with $1/L$. The extrapolation to $1/L \to 0$ gives $\beta(\infty) = 0.126$ $[T < T_c(L)]$ and $\beta'(\infty) = 0.868$ $[T > T_c(L)]$.

As a third estimation, the slopes obtained from the log–log plots of the scaling relations for the maxima of the specific heat and the susceptibility at their critical temperatures $T_c(L)$ are used. The slopes of the straight lines give $\alpha/\nu = 0.035$, and $\gamma/\nu = 1.85$. Compared to the scaling relations at $T_c(\infty)$, these scaling relations improve α/ν and γ/ν toward the theoretical values.

As a fourth way of estimation, the critical exponents $\beta(L)$ and $\gamma(L)$ for each lattice are obtained from the log–log plots of the following relations:

$$M \propto [T - T_c(L)]^{\beta} \tag{5}$$

$$kT\chi \propto [T - T_c(L)]^{-\gamma} \tag{6}$$

The critical exponents for each lattice are computed by using the data within the interval $0.015 < T - T_c(L)/T_c(L) < 0.15$, and a best fit to straight lines. These critical exponents are plotted against $1/L$ (Figs. 3 and 4). The

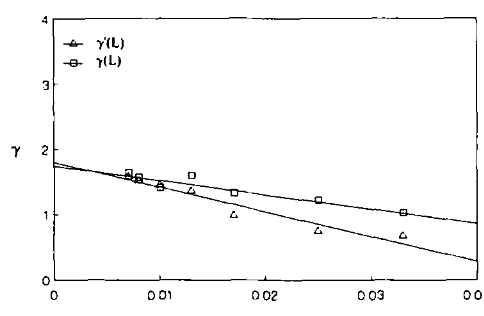

Fig. 4. Variation of the critical exponent $\gamma(L)$ for the susceptibility with $1/L$. The extrapolation to $1/L \to 0$ gives $\gamma(\infty) = 1.75$ $[T < T_c(L)]$ and $\gamma'(\infty) = 1.79$ $[T > T_c(L)]$.

data lie on straight lines, and their extrapolations to $1/L \rightarrow 0$ give $\beta = 0.126$ (0.868) and $\gamma = 1.75$ (1.79) for $T < T_c(L)$ [$T > T_c(L)$]. These values are in very good agreement with the theoretical ones. An overall error of about 3% is estimated for the values of the critical exponents.

4. CONCLUSION

In this note the static critical exponents α, β, γ for the two-dimensional Ising model have been computed on a cellular automaton by using the algorithm due to Creutz. The data are analyzed according to the finite-size scaling theory. Several independent estimations for these critical exponents within this theoretical framework are in good agreement with each other and with their theoretical values within the error limits. However, the critical exponent α for the specific heat deserves a separate treatment.

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